MR Image Encoding by Spatially Selective rf Excitation: An Analysis Using Linear Response Models

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ABSTRACT: Spatially selective radiofrequency (rf) excitation has been applied to encode magnetic resonance (MR) images using non-Fourier basis sets such as wavelets. While non-Fourier-encoded MRI has the advantage of flexibility allowing for various adaptive imaging strategies, pulse sequence implementation for spatially selective rf excitation presents unique problems. Two linear response models are described that aid in the analysis of non-Fourier-encoding methods. The models represent important characteristics of the physical system involved in the encoding process and formalize mathematically aspects of the implementation that are relevant for pulse sequence design. As an illustrative example, these response models are applied for an analysis of a three-dimensional echo-planar method that encodes MR images using a basis derived by singular value decomposition. © 1999 John Wiley & Sons, Inc. Int J Imaging Syst Technol, 10, 143–150, 1999

I. INTRODUCTION
The technique of spatially selective radiofrequency (rf) excitation (Morns, 1986) is well known in magnetic resonance imaging (MRI) and is used routinely for the imaging of limited volumes (slices). It is not, however, generally considered as a technique for encoding images although early efforts with Hadamard encoding were made along these lines (Oh et al., 1984; Cho et al., 1987; Bolinger and Leigh, 1988). More recently, spatially selective rf excitation has been used to implement non-Fourier–encoding techniques such as wavelet encoding (Panych et al., 1993, 1998; Oshio et al., 1993; Wendt et al., 1996; Peters and Wood, 1996; Gelman and Wood, 1996, 1998) and encoding by singular value decomposition (SVD) (Zientara et al., 1994; Panych et al., 1996a). These non-Fourier approaches are suited to the application of adaptive imaging approaches (Zientara et al., 1994; Panych and Jolesz, 1994) which aim to dynamically tailor the data acquisition strategy and exploit all available information about the field-of-view contents.

In recent work implementing SVD encoding (Panych et al., 1996a), we introduced an input–output response model to describe the SVD encoding of spins by spatially selective rf excitation. The response interpretation of rf excitation is complementary to the $k$-space description of small-flip-angle excitations (Pauly et al., 1989; Reese and Pearlman, 1994). In addition, however, it incorporates some effects due to the physical process of rf excitation. Because the linear response formalism gives both a mathematical as well as a partial physical model, it is especially suited for adaptive encoding methods. This is especially true for the SVD approach where success of the data-encoding strategy depends on an accurate estimate of what signal response will be obtained when a specific rf excitation is used.

In this article, this system response approach is outlined in detail. The formalism, both for understanding and for implementing encoding methods based on the use of spatial encoding by manipulation of spatially selective rf profiles, is presented. We demonstrate the utility of the approach in the design of three-dimensional (3D) SVD encoding methods using a pulse sequence with multidimensional spatially selective rf pulses and interleaved echo-planar pulse trains.

II. ENCODING BY SELECTIVE EXCITATION
In previous work involving non-Fourier encoding, a spatial domain description of the procedure is used in which 1D profiles shaped like the functions of some alternative non-Fourier basis set (such as Hadamard functions or wavelets) are excited across the field of view (Oh et al., 1984; Weaver et al., 1992; Panych et al., 1993). In this section, this description of MR image encoding using selective excitation is briefly reviewed. For simplicity, we describe a simple 3D imaging case in which non-Fourier encoding is performed only along what would otherwise be the slice direction and Fourier (phase and frequency) encoding is applied in-plane. An adaptation of line-scan imaging (Ernst, 1987) where non-Fourier encoding is done along what would otherwise be the phase-encoding (Kumar et al., 1975) direction, is also briefly discussed.

Full 3D encoded Fourier transform MRI is not as commonly used in clinical imaging as is the selective excitation of thin sections or slices (Morns, 1986). In slice-selective imaging, Fourier encoding is applied in only two dimensions ($x$ and $y$) with spatially selective excitation being implemented in the third dimension ($z$). The acquired data, $S(k_x, k_y, z_o)$, for such a case can be expressed by the integral

$$S(k_x, k_y, z_o) = \int \int \int s(x, y, z) P(z - z_o) e^{-i(k_xx + k_yy)} \, dx \, dy \, dz$$

(1)
where $P(z - z_o)$ is the excitation profile in the $z$ dimension, $s(x, y, z)$ is an MR density function that we aim (at least partially) to image in some fashion, and $k_x$ and $k_y$ are parameters dependent on $x$ and $y$ gradient pulse amplitudes and duration. Ideally, $P(z - z_o)$ is close to zero everywhere except in a narrow interval $(\pm \Delta z)$ about the location $z = z_o$. Inverting Equation (1) by Fourier transformation with respect to $k_x$ and $k_y$ gives

$$s_s(x, y) = \int s(x, y, z) P(z - z_o) \, dz$$  \hspace{1cm} (2)$$

where $s_s(x, y)$ is the reconstructed MR signal density function integrated over the $z$ direction, therefore reflecting the complete contents of the slice.

It is possible in theory to excite a spatial profile of arbitrary shape by appropriate shapping of the modulation envelope of the rf excitation pulse. If these spatial excitation profiles are functions of a non-Fourier orthogonal basis such as wavelets, then the field of view is encoded along that basis and can be reconstructed by appropriate inverse transformation (assuming a complete set of encoding functions is used). For low-flip-angle excitations, designing the excitation profile is as simple as making the rf pulse envelope equal to the Fourier transform of the desired profile (Hoult, 1979).

At larger flip angles, the nonlinearity of the Bloch equations complicates rf pulse design (Pauly et al., 1991); however, we have shown that encoding with higher-flip-angle pulses is possible using a simple iterative design method (Kyriakos and Panych, 1997).

The line scan technique (Ernst, 1987) can be used to further restrict the imaged volume by employing a spatially selective excitation pulse that is in a dimension ($y$) orthogonal to a slice-selective 180° refocusing pulse. Similarly to the multislice adaptation, if one excites a set of line excitations profiles that are functions of a non-Fourier orthogonal basis, then the field of view is encoded along $y$ in that basis and can be reconstructed by inverse transformation. A modified line scan approach has been used for both wavelet (Panych et al., 1993; Oshio and Wood, 1996) and SVD-encoded MRI (Zientara et al., 1994; Panych et al., 1996a). The modified multislice approach has been used previously for wavelet encoding (Gelman and Wood, 1996, 1998).

III. LINEAR RESPONSE MODEL OF SELECTIVE EXCITATION

A. rf-Selective Excitation Profile. The spatially selective rf pulse, $p(t)$, from a typical slice-selective sequence is shown in Figure 1 as it is usually implemented—that is, as a stepwise approximation to a desired continuous shape, such as a sinc-shaped function. Let $p(t)$ be represented by a hard-pulse approximation (Subramanian et al., 1985) (as shown in Figure 1) such that the energy delivered by $p(t)$ from time $t$ to $t + \Delta t_p$ is delivered by an impulse at time $t$. In this approximation, spins are flipped instantaneously at time $t$ and undergo free precession in the time interval $\Delta t_p$. The hard-pulse train, $p_H(t)$, will be represented by the series

$$p_H(t) = \sum_n p_n \delta(t - n\Delta t_p)$$  \hspace{1cm} (3)$$

where $p_n$ is a complex number representing both the amplitude and the phase of the $n$th hard pulse. Assume that $s(x, y, z)$ is the signal density when all spins have been flipped 90° and are in phase and that the amplitude and phase of $\{p_n\}$ are referenced to the amplitude and phase of the pulse that produces $s(x, y, z)$. The total signal from all spins at time $\tau$ due to the $n$th hard pulse can be written as

$$S(k_x, k_y, k_z) = \int \int s(x, y, z) \{p_n e^{-i\omega t}e^{-i(k_xx + k_yy + k_zz)} \, dx \, dy \, dz$$  \hspace{1cm} (4)$$

where $k_n = \gamma G_x \Delta t_p$ and where $k_x$ and $k_y$ are parameters dependent on $x$ and $y$ gradient pulse amplitudes and duration. The above equation is valid for a hard pulse of any flip angle.

For small flip angles, the response to any arbitrary hard-pulse train is a superposition of the responses to the individual pulses making up the pulse train. The principle of superposition is a fundamental property of linear systems and, in fact, can be said to define this class of systems (Oppenheim and Schafer, pp. 480–484). Flip angles are small enough for superposition to be valid as long as the longitudinal magnetization is not appreciably disturbed when excited by the hard-pulse train (Hoult, 1979). In other words, it is assumed that the magnetization has not been tilted far from the longitudinal axis, and therefore $\sin(\alpha) \equiv \alpha$, where $\alpha$ is the flip angle.

Applying superposition in the case of excitation of the system by the hard-pulse train, the total signal is

$$S(k_x, k_y) = \int \int s(x, y, z) \left[ \sum_n p_n e^{-i\omega t} \right] e^{-i(k_xx + k_yy)} \, dx \, dy \, dz$$  \hspace{1cm} (5)$$

The summation term in brackets in Equation (5) is the excitation profile and is equal to the discrete Fourier transform of the series $\{p_n\}$ that defines the hard pulse train. In the limit of infinitesimal
Figure 2. Radiofrequency pulses that can be used to excite the MR system. The second two composite pulses are the sum (c₁(t)) and difference (c₄(t)) of component pulses, c₁(t) and c₄(t). The theoretical responses to the composite pulses can be obtained by computing the sum and difference of the responses to c₁(t) and c₄(t).

Let an rf pulse \( p(t) \) be represented by a linear combination of unit box rf pulses, \( \{ \Pi_n(t) \} \), such that

\[
p(t) = \sum_n p_n \Pi(t - n\Delta t_p) = \sum_n p_n \Pi_n(t)
\]

(7)

where

\[
\Pi(t) = 1, \quad 0 \leq t \leq \Delta t_p
\]

\[
= 0, \quad \text{otherwise}
\]

(8)

The complex-valued coefficients \( \{ p_n \} \) define the amplitude and the phase for each of the unit box rf pulses that comprise \( p(t) \).

The general echo response \( y(t) \) to an rf excitation is sampled by the imaging system to give a complex-valued sequence \( \{ y_k \} \) such that

\[
y_k = \int y(t) \Lambda(t - k\Delta t_s) dt = \int y(t) \Lambda_k(t) dt
\]

(9)

where \( \Lambda_k(t) \) is a temporal sampling function that samples the echo in a window of width \( \Delta t_s \) in the time interval between \( k\Delta t_s \) and \( (k + 1)\Delta t_s \).

Let the echo response \( R_n(t) \) to an rf unit box pulse \( \Pi_n(t) \) be sampled by the imaging system to give a complex-valued sequence \( \{ R_{n,k} \} \) such that

\[
R_{n,k} = \int R_n(t) \Lambda_k(t) dt
\]

(10)

Applying superposition, the echo response, \( y(t) \), to an rf pulse, \( p(t) \), is equal to a superposition of the responses to the individual component rf box pulses making up \( p(t) \)

\[
y(t) = \sum_n p_n R_n(t)
\]

(11)

Multiplying both sides of Equation (11) with the sampling function and integrating

\[
\int y(t) \Lambda_k(t) dt = \int \sum_n p_n R_n(t) \Lambda_k(t) dt
\]

(12)

and substituting from Equations (9) and (10),

\[
y_k = \sum_n p_n R_{n,k}
\]

(13)

Then, defining \( \tilde{y} \) as a vector with elements \( y_k \), \( \tilde{p} \) as a vector with elements \( p_n \), and \( \tilde{R} \) as a matrix whose \( n \)th row contains the sampled echo response after excitation by \( \Pi_n(t) \), a matrix form for the superposition response can be written as

\[
\tilde{y} = \tilde{p} \tilde{R}\tilde{n}
\]

(14)
Thus, if \( \tilde{R}_c \) is known, then the response to any rf excitation that is a superposition of unit box pulses (as in Figure 1) can be computed using Equation (14). In this sense, \( \tilde{R}_c \) is the system response matrix.

By definition, each row of \( \tilde{R}_c \) contains the sampled echo response to an rf unit box excitation, where the \( \text{th} \) row contains the response to \( \Pi_k \). The elements of the vector representing \( \Pi_k \) are equal to those of the kronecker delta sequence \( \delta_n \) (i.e., all zeros except in the \( \text{th} \) position). This is a digital impulse function, and in this sense, \( \tilde{R}_c \) is a system impulse response matrix.

**C. The \( k \)-Space Response Model.** Spatially selective rf excitation as an encoding method can be understood through the interpretation of Equation (14). The key to this understanding is the mapping between \( \tilde{R}_c \) and the \( k \)-space matrix \( \tilde{S}_k \). To see this, consider the sequence shown in Figure 3, which can be used to acquire Fourier-encoded image data in two dimensions. (The signal is integrated over the third dimension.) It differs from a standard phase-encoding sequence in that the phase-encoding gradient is kept constant and the position of a small-flip-angle box pulse, \( \Pi_{\text{w},r} \), is shifted by \( n \Delta t \), to vary the dephasing for each encode. At the \( \text{th} \) excitation, the acquired data samples are placed into the \( \text{th} \) row of \( \tilde{S}_k \) and represent a new line in \( k \)-space. When Fourier transformed, \( \tilde{S}_k \) gives an image array.

According to the previous section, the acquired data samples from an excitation by each of the box pulses also represents a different row in the system response matrix \( \tilde{R}_c \). Thus, the data obtained when using \( \{ \Pi_{\text{w}} \} \) give both the \( k \)-space matrix and the response matrix of the system. In this case, therefore, there is a direct mapping between \( \tilde{R}_c \) and \( \tilde{S}_k \).

Consider now the pulse sequence shown in Figure 4, which is used for spatially selective excitation. The rf pulses are linear combinations of the rf unit box pulses. Except for the rf inputs, the two sequences in Figures 3 and 4 are identical. Therefore, from the systems point of view, these sequences are identical.

Image encoding using the spatially selective pulse sequence shown in Figure 4 is a matter of determining \( \tilde{R}_{\Pi} \). Let \( \tilde{P} \) be a matrix whose rows represent different rf pulse excitations and \( \tilde{Y} \) be a matrix whose rows contain the sampled echo response to pulses defined by each of the coinciding rows of \( \tilde{P} \). Therefore, Equation (14) can be rewritten as

\[
\tilde{Y} = \tilde{P} \tilde{R}_{\Pi} \tag{15}
\]

If the set of excitation pulses is the set of unit box pulses, \( \{ \Pi_{\text{w}} \} \), then \( \tilde{P} \) is the identity matrix, \( \tilde{Y} \), and the output matrix is equal to \( \tilde{R}_{\Pi} \). In this case, since \( \tilde{R}_{\Pi} = \tilde{S}_k \), no further operation other than 2D Fourier transformation of \( \tilde{Y} \) is required to reconstruct the image.

In general,

\[
\tilde{Y} = \tilde{P} \tilde{R}_{\Pi}
\]

To reconstruct an image using spatially selective excitation with a set of pulses defined by \( \tilde{P} \), it is necessary that \( \tilde{P} \) be invertible. Ignoring signal-to-noise ratio differences, all orthogonal matrices are equally acceptable. (Nonorthogonal matrices of full rank are also acceptable as long as their condition number is not too high.) The identity matrix is the orthogonal matrix that can be said to produce Fourier encoding since each excitation gives a new line in \( k \)-space. All other matrices give combinations of \( k \)-space lines and produce non-Fourier encoding. Whether Fourier or non-Fourier encoding is used, \( \tilde{S}_k \) can be obtained; therefore, we will refer to this as a \( k \)-space response model.

**D. The General Response Model.** In Equation (15), each row vector of \( \tilde{R}_{\Pi} \) contains the response to the excitation by a unit box rf pulse. However, the interpretation of Equation (15) can be extended such that these row vectors represent the responses to any convenient set of component rf pulses, \( \{ c_n(t) \} \). A general response model involving a set of such pulses is outlined in this section.

Suppose we have an imaging pulse sequence with a rf section that excites a basic volume element and that the rf waveform can be altered to place the volume element anywhere within the imaging volume. Suppose further that the rf excitation pulse \( c_n(t) \) that excites a basic volume element is followed by acquisition of a set of phase- and frequency-encoded data values. Assume also that Fourier reconstruction of the data acquired from one or several rf excitations will give an image of the contents of a basic volume element. Taken together, the data reconstructed from all of the basic volume elements represent a volume of interest.

A general input–output response matrix \( \tilde{R}_c \) is constructed by placing the complex-valued data samples acquired from each of the component volume elements into a separate row vector of \( \tilde{R}_c \) so that the matrix contains the response to each of the rf pulses that excite the basic volume elements. By superposition, the response to any linear combination of the rf pulses can then be computed from a linear combination of the responses recorded in the row vectors of \( \tilde{R}_c \). Mathematically, if \( \tilde{P} \) represents the coefficients of the linear combination, then the computed response is just \( \tilde{P}^T \tilde{R}_c \). If \( \tilde{P}_c \) is a matrix where each row vector defines a linear combination of rf pulses that excite the component volumes, then

\[
\tilde{Y} = \tilde{P}_c \tilde{R}_c \tag{17}
\]

Where the row vectors of \( \tilde{Y} \) contain the responses to each component rf pulse. By definition, \( \tilde{R}_c \) contains the information necessary to reconstruct images of all the component volumes. This information can be reconstructed from \( \tilde{Y} \) as long as \( \tilde{P}_c \) is invertible.

![Figure 3](image3.png)

**Figure 3.** A multishot gradient-echo sequence for phase encoding. The phase-encoding gradient strength is constant for all encodes, but the position of the rf unit box pulse is shifted by \( \Delta t \), on each of \( N \) encodes. This sequence is not slice selective.

![Figure 4](image4.png)

**Figure 4.** Multishot gradient-echo sequence for spatial encoding. On each excitation, a different rf unit box pulse train is used (one pulse train is shown in the figure). This sequence is not slice selective.
Equation (17) generalizes for all basic volume elements the multislice encoding principles first introduced by Oh et al. (1984). Oh and colleagues referred to \( \hat{P} \) as a “multislice encoding matrix.” Such a matrix is most often used for Hadamard encoding of slices. In these approaches, rf pulses are designed to excite linear combinations of slices as defined by the multislice encoding matrix, usually with the goal of increasing the signal-to-noise ratio. Data are stored in a result matrix or matrices with a phase-encoded line or lines from multiple excitations in each of the rows. Decoding the slice combinations is achieved by multiplication of the result matrix with the inverse of the multislice encoding matrix.

If each of the component rf pulses \( c_n(t) \) used to excite the basic volume element is represented by a row vector in the matrix \( \tilde{C} \) such that each row vector defines the pulse in terms of the unit box rf pulses, then

\[
\tilde{R}_c = \tilde{C} \tilde{R}_H
\]  

Equation (18) merely says that the response to each component rf pulse \( c_n(t) \) can also be computed from the responses to the individual box rf pulses. Combining Equations (17) and (18),

\[
\tilde{Y} = (\hat{P}_c \tilde{C}) \tilde{R}_H
\]

where each row vector of \( (\hat{P}_c \tilde{C}) \) defines a combination rf pulse in terms of its unit rf box components. If \( (\hat{P}_c \tilde{C}) \) in Equation (19) is invertible, then \( \tilde{R}_H \) can also be recovered. Recall from the previous section that \( \tilde{R}_H \) maps to a k-space matrix. In this case, the general response model and the k-space response models can both be used to recover k-space information using spatially selective rf encoding. In general, however, \( (\hat{P}_c \tilde{C}) \) is not invertible.

In the case of standard wavelet encoding, \( \hat{P}_c \) is a wavelet matrix operator and \( \tilde{C} \) contains definitions in terms of the unit box rf pulses of the rf pulses that excite scaling function-shaped profiles. Although \( \hat{P}_c \) is invertible, \( \hat{P}_c \tilde{C} \) is not in general. Thus, \( \tilde{R}_c \) can be recovered by standard wavelet encoding but the k-space information represented by \( \tilde{R}_H \) can not generally be recovered. This reflects the fact that wavelet and Fourier function spaces do not completely overlap for sets of functions at the same level of resolution. Note that in Equation (15), we could choose \( \hat{P} \) equal to a wavelet matrix operator and then recover \( \tilde{R}_H \). As discussed in Panych et al. (1998), however, this only represents wavelet encoding in a digital sense because we would not actually have excited wavelet-shaped profiles as is done in standard wavelet encoding.

**IV. SVD-ENCODED MRI**

**A. Deriving Optimal rf-Encoding Vectors Using the SVD.**

Previously, we described a new method of encoding MR images based on singular value decomposition (SVD) (Zientara et al., 1994). The method was proposed for use in dynamic MRI to reduce the acquisition of redundant information. In that method, the SVD of a matrix containing prior image data was used to choose spatially selective rf excitation profiles for encoding subsequent images. We claimed that because of the linear algebraic nature of the SVD, this choice of encoding profiles would be near optimal in the sense of requiring the minimum number of encoding steps for dynamically updating the image estimate. A fast gradient-echo method (Panych et al., 1996a) was later implemented and used to obtain subsecond 2D image updates of the events following contrast agent injection into a phantom. More recently, we reported the implementation of a 3D-interleaved echo-planar (EP) technique using SVD encoding (Panych et al., 1996b).

Singular value decomposition encoding can be understood from the k-space response model [see Equation (15)]. To begin the SVD-encoding process, the identity matrix \( I \) is chosen as the initial rf-encoding matrix \( \hat{P} \). In other words, a set of \( N \) rf unit box rf pulses, \( \{ \Pi_n(t) \} \), is used and the system impulse response \( \tilde{R}_H \) is acquired after \( N \) rf excitations. The system autocorrelation matrix \( \tilde{R}_H \tilde{R}_H^\dagger \) is then computed and diagonalized,

\[
\tilde{R}_H \tilde{R}_H^\dagger = \tilde{V} \tilde{D} \tilde{V}^\dagger
\]

The columns of \( \tilde{V} \) reveal the natural modes or eigenmodes of the linear system ordered according to which are most dominant.

After diagonalization, \( \tilde{V}^\dagger \), the matrix containing the \( L \) eigenvectors of \( \tilde{V} \) with the \( L \) largest associated eigenvalues, is chosen as the rf-encoding matrix for updating the response matrix in the dynamic imaging phase. It is assumed that the response to the \( L \leq N \) dominant eigenmode excitations is sufficient to reconstruct the system response to some specified accuracy. The full updated system response matrix \( \tilde{R}_H \) is reconstructed from the \( L \) output vectors by matrix multiplication with \( \tilde{V}_l \) (a procedure analogous with zero-filling k-space matrices). As discussed previously, because \( \tilde{R}_H \) maps to a k-space matrix, image updates can be reconstructed by Fourier transformation of \( \tilde{R}_H \).

The simplest approach is to perform the diagonalization one time only, at the beginning of the imaging session, and then to use the same eigenvectors throughout the dynamic imaging phase. This approach was used by us for following contrast changes with a fast gradient-echo SVD method (Panych et al., 1996a). The implicit assumption behind this approach is that the basic eigenstructure remains intact and that only the system eigenvalues undergo changes. For contrast variations, it appears that this approach works well. For some applications, however, a more general approach is needed that does not rely on the assumption of a static eigenstructure. In other words, the eigenvectors must be updated throughout the dynamic phase of imaging. Such an approach has been implemented in a fluoroscopic mode for use in interventional MRI (Zientara et al., 1997a).

**B. A 3D SVD-Encoding Example: k-Space Response Model.** In this and the following section, we will outline two novel 3D imaging methods based on SVD encoding. Both methods involve derivation of optimal rf-encoding vectors using 3D image sets. The two methods use the same basic pulse sequence design, which is shown schematically in Figure 5. The pulse sequence consists of a 2D rf excitation portion followed by multiple gradient echoes for phase encoding. In general, a sequence with \( \sqrt{N} \) rf sections and an echo train length of \( \sqrt{N} \) can be used to encode an \( N \times N \times N \) image with \( N \sqrt{N} \) excitations. The sequence in Figure 5 is shown for simplicity’s sake with only four rf and four acquisition sections.

The first 3D SVD-encoding method, based on the k-space response model, is described as follows for the simple \( N = 16 \) case. The rf pulse-labeled \( r_{f1} \) in Figure 5 is a typical composite rf pulse that might be used for 3D SVD encoding. The set of unit box rf pulses, shown schematically by \( r_{f2} \), forms the basis for the composite SVD pulses. The individual box pulses can be used to encode 3D k-space as described by the trajectories shown in Figure 6 and discussed below.

Assume that the first excitation is with the unit box pulse in the A group at the leftmost position. Each of the echoes recorded at \( S_1 \),
S_2, S_3, and S_4 fills a separate line in k-space as shown by the set of 16 × 4 points labeled by A_1 in Figure 6. Each of the unit box pulses in the A group fills several lines in k-space offset along k_x as shown for three pulses (A_1, A_8, and A_16) in the figure. The trajectory through the points filled by excitation with the B_1, C_1, and D_1 pulses are also shown in the figure. After 16 × 4 excitations with all pulses from all four groups, the 16 × 16 × 16 k-space array will be filled. In general, after \( N \sqrt{N} \) excitations, the \( N^3 \) k-space array will be filled.

The \( N^3 \) k-space array maps to an \( (N\sqrt{N}) \times (N\sqrt{N}) \) system response matrix as follows. Again, assume that the unit box pulses as shown in rf_2 are used. Each set of four echoes acquired \((S_1-S_4)\) on each excitation comprises a single row (of 16 × 4 elements) in the system impulse response matrix. The full 64 × 64 impulse response matrix is obtained after 64 excitations. In general, an \((N\sqrt{N}) \times (N\sqrt{N})\) response matrix is obtained after acquiring \( \sqrt{N} \) echoes on each of \( \sqrt{N} \) excitations.

For dynamic SVD imaging, the procedure to determine the optimal rf pulse excitations is the same as that described in the previous section. After acquiring the system impulse response matrix (i.e., the baseline data), the system autocorrelation matrix is diagonalized and only those vectors with the highest eigenvalues are selected to define the composite pulses (rf_) used for rf encoding in the dynamic phase. It has been found that the response matrix may be reconstructed with as few as \( N \) eigenmode excitations, and in an implementation on a 1.5-T Signa system with standard gradients, 64^3 image updates could be obtained in as little as 5–6 s (Panych et al., 1996b).

C. A 3D SVD-Encoding Example: General Response Model. The general response model can be used to design an alternative 3D SVD-encoding approach using the pulse sequence in Figure 5. A version of this sequence was first proposed for Fourier spliced-pencil imaging, a very fast single-shot method to image 2D strips (Heid, 1995). The individual rf sections consisted of Gaussian-shaped pulses with a Gaussian weighting over all the rf sections. Thin 2D strips can be imaged with this sequence with the positioning of the strips controlled by manipulation of the rf pulse waveforms.

If the data acquired from each strip are placed in the row of a matrix, a general response matrix \( \bar{R}_c \) is obtained. The general response model can then be used to design 3D SVD encoding with planar strips. With an initial estimate of \( \bar{R}_c \), the near-optimal rf pulses for encoding a set of component volume elements can be found using the SVD approach. Diagonalization of \( \bar{R}_c \) gives the eigenvectors of the response matrix and their relative weights. The rf pulse that will excite a specific combination of volume elements represented by an eigenvector \( \bar{v} \) is found from the vector matrix product \( \bar{P}_c \bar{C} \), where \( \bar{P}_c \) and \( \bar{C} \) are as previously defined. Updated estimates of \( \bar{R}_c \) are obtained by reconstructing results from the eigenmode excitations.

Note that the result is different from that obtained using the 3D SVD method outlined in the previous section, which is based on the k-space response model. The previously described method derives optimal combinations of k-space encodes, and ideally, the reconstructed images should look the same as if unit box rf pulses were used for phase encoding. The multistrip SVD-based method derives optimal combinations of planar strips, and ideally, the reconstructed images should look the same as if the strips were imaged individually. In the multistrip method, the data will not generally be isotropic because encoding in the third dimension depends on the strip thickness and its excitation profile along that dimension.

A simulation of the 3D SVD-encoding method based on the general response model demonstrates the data reduction possible when encoding with a reduced set of eigenvectors. The first step of

Figure 5. Pulse sequence schematic for 3D-SVD encoding showing four rf sections and four multiple gradient echoes. For general \( N^2 \) encoding, there are \( \sqrt{N} \) rf sections and \( \sqrt{N} \) gradient echoes. Each rf section is a composite pulse of \( N \) unit box pulses. Each gradient echo is sampled \( N \) times.

Figure 6. k-space points encoded by the 3D pulse sequence in Figure 5 when the set of shifted unit box pulses (rf_) are used.
the response matrix with the reduced eigenvector matrix ($R_{n}$) encoding. Encoded data are simulated by matrix multiplication of $R_{n}$ to 16 of the eigenvectors. (Bottom right) A Fourier-encoded image obtained by simulating reconstruction using only $\frac{1}{5}$ of the eigenvectors. (Bottom left) The corresponding coronal section obtained by simulating reconstruction using only $\frac{1}{5}$ of the eigenvectors. For whichever rf implementation is chosen, diagonalizing the system autocorrelation matrix is guaranteed to uncover the optimal pulse shapes to apply in the implementation, assuming that the system eigenstructure remains relatively intact. [An in-depth discussion of the suitability of the assumption of optimality can be found in Zientara et al. (1998a). In general, expressing the imaging operation in terms of an input-output response matrix allows us, regardless of implementation details, to draw on a variety of new linear-algebraic methods for near optimal encoding that are currently under investigation (Zientara et al., 1997b, 1998b).

Applying linear systems approaches to MRI encoding may have significant potential for future development of adaptive and optimal imaging strategies. Casting imaging within the linear systems framework automatically raises the possibility of applying systems engineering approaches to the dynamic imaging problem. In SVD encoding, for example, analysis of the system response function allows for identification of system eigenmodes and suggests optimal excitation strategies. Adaptive methods which have been previously developed to deal with time-varying linear systems may be applied to track the changing eigenmodes (Moonen et al., 1992; DeGroat and Roberts, 1988) of the MR system and to select optimal waveforms (Tewfik et al., 1994) for rf excitation. The linear systems framework serves to generalize the imaging problem and thereby suggest solutions that might not otherwise seem apparent when using traditional spatial domain or $k$-space interpretations.

Feedback control was used previously by others to dynamically control, by rf pulse manipulation, the state of the nuclear magnetization in a homogeneous sample and to make it track a predetermined reference (Schiano et al., 1991, 1992). In dynamic imaging, the ultimate challenge is to track the state of the magnetization throughout a complex volume undergoing constant changes. Meeting this challenge requires both effective imaging models and robust adaptive algorithms to drive the encoding process.

Work continues in our laboratory in the extension of system modeling approaches. We are actively pursuing the development of perturbation approaches to include the effects of nonlinearities in our models (Kyriakos and Panych, 1997). We have also made attempts to characterize the transient effects of variable $T_1$ weight-

V. DISCUSSION

In this article, an analysis of encoding methods that employ spatially selective rf excitation is presented. The analysis is based on the fact that the response to an arbitrary small flip-angle rf excitation pulse is equal a superposition of responses to individual component pulses comprising the composite pulse. Spatially selective rf pulses are defined in terms of a set of building-block unit box rf pulses that can be used for $k$-space encoding. The response to a set of these box pulses comprises the impulse response of the system which is represented by a system response matrix. It was shown that the system response matrix maps to the $k$-space matrix, a fact we have used to define a $k$-space response model for encoding images using rf pulses. A general response model is also presented whereby the impulse response is generalized as the response to individual rf pulses that excite component volume elements.

One difference between our approach and earlier work with multislice encoding by rf pulse manipulation is the extension to include the encoding of all basic elements that can be selected by rf pulse manipulations, whether they are $k$-space locations or volume units. A second and more important difference, however, is the interpretation in terms of system response theory specifically through identification of the response matrix. The system response matrix incorporates the effect of relaxation and off-resonance evolution on the rf excitation, and this is independent of the actual interpretation of the system response matrix with respect to $k$-space sampling or the excitation of spatial profiles. This is very important for adaptive approaches such as the SVD-encoding method, where an optimal choice of the encoding vectors depends completely on an accurate characterization of the system. In the approaches described here, rf pulses are selected in terms of what maximally excites the system as it is defined by the system response matrix.

For whichever rf implementation is chosen, diagonalizing the system autocorrelation matrix is guaranteed to uncover the optimal pulse shapes to apply in the implementation, assuming that the system eigenstructure remains relatively intact. [An in-depth discussion of the suitability of the assumption of optimality can be found in Zientara et al. (1998a). In general, expressing the imaging operation in terms of an input-output response matrix allows us, regardless of implementation details, to draw on a variety of new linear-algebraic methods for near optimal encoding that are currently under investigation (Zientara et al., 1997b, 1998b).

Applying linear systems approaches to MRI encoding may have significant potential for future development of adaptive and optimal imaging strategies. Casting imaging within the linear systems framework automatically raises the possibility of applying systems engineering approaches to the dynamic imaging problem. In SVD encoding, for example, analysis of the system response function allows for identification of system eigenmodes and suggests optimal excitation strategies. Adaptive methods which have been previously developed to deal with time-varying linear systems may be applied to track the changing eigenmodes (Moonen et al., 1992; DeGroat and Roberts, 1988) of the MR system and to select optimal waveforms (Tewfik et al., 1994) for rf excitation. The linear systems framework serves to generalize the imaging problem and thereby suggest solutions that might not otherwise seem apparent when using traditional spatial domain or $k$-space interpretations.

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Figure 7. (Top left) One coronal section from a 3D brain data set. (Top right) The corresponding coronal section obtained by simulating an SVD image reconstruction using only $\frac{1}{5}$ of the eigenvectors. (Bottom left) The corresponding coronal section obtained by simulating reconstruction using only $\frac{1}{5}$ of the eigenvectors. (Bottom right) A Fourier-encoded image obtained by simulating reconstruction from $\frac{1}{5}$ of the $k$-space data.
ing (Saiviroonporn et al., 1997; Saiviroonporn, 1997). Further work is planned in the development of stochastic and multiparametric models. In addition, models that incorporate gradient response are also needed along with methods to incorporate system constraints. We believe that only when these models are developed and formalized mathematically and when computer hardware to enable on-line computation of encoding parameters based on these models is available will the full potential of adaptive imaging based on spatially selective rf encoding be realized.

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